Three Dimensional Strongly Symmetric Circulant Tensors

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Abstract

In this paper, we give a necessary and sufficient condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite. In some cases, we show that this condition is also sufficient for this tensor to be sum-of-squares. Numerical tests indicate that this is also true in the other cases.

Key words: H-eigenvalue, strongly symmetric tensor, circulant tensor, sum of squares, positive semi-definiteness.

AMS subject classifications (2010): 15A18; 15A69

1 Introduction

Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be an even order n dimensional real symmetric tensor, i.e., $m = 2k$, $i_1, \cdots, i_m = 1, \cdots, n$, $a_{i_1\cdots i_m}$ are real and invariant under any index permutation. Then $\mathcal{A}$ is corresponding to a homogeneous polynomial $f(\mathbf{x})$ for $\mathbf{x} \in \Re^n$, defined by

$$f(\mathbf{x}) = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}.$$

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If $f(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Re^n$, then $\mathcal{A}$ is called a positive semi-definite (PSD) tensor, f is called a PSD or nonnegative polynomial. To identify a PSD symmetric tensor or a PSD homogeneous polynomial is an important problem in theory and practice [1, 10, 22]. It is NP-hard to identify a general even order symmetric tensor is PSD or not. Recently, it was discovered that several easily checkable classes of special even order symmetric tensors are PSD. These include even order symmetric diagonally dominated tensors [22], even order symmetric B_0 tensors [24], even order Hilbert tensors [27], even order symmetric M tensors [30], even order symmetric double B_0 tensors [14], even order symmetric strong H tensors [15, 11], even order strong Hankel tensors [23], even order positive Cauchy tensors [4], etc.

Hankel tensors arise from signal processing and data fitting [2, 8, 21, 23, 29]. They are symmetric tensors. It is known that even order strong Hankel tensors are PSD. A Hankel tensor is called a strong Hankel tensor if its generating vector generates a PSD Hankel matrix. This condition is checkable. However, there is still no checkable condition for verifying a general even order Hankel tensor is PSD or not. In [17], it was proved that an even order strong Hankel tensor is a sum-of-squares (SOS) tensor. It was also discovered there that there are PSD Hankel tensors, which are not strong Hankel tensors, but still SOS tensors. Thus, an open question is raised in [17]: are all PSD Hankel tensors SOS tensors? If the answer to this question is "yes", then the problem for determining a given even order Hankel tensor is PSD or not can be solved by solving a semi-definite linear programming problem [17, 12, 13].

It is not easy to answer the question raised in [17]. For general homogeneous polynomials, this problem was first studied by Hilbert [10]. In 1888, Hilbert [10] proved that only in the following three cases, a PSD homogeneous polynomial, of degree m and n variables, definitely is an SOS polynomial: 1) $m = 2$; 2) $n = 2$; 3) $m = 4$ and $n = 3$, where m is the degree of the polynomial and n is the number of variables. Hilbert proved that in all the other possible combinations of n and even m, there are PSD non-SOS (PNS) homogeneous polynomials. However, Hilbert did not give an explicit example of PNS homogeneous polynomials. The first explicit example of PNS homogeneous polynomials was given by Motzkin [20] in 1967. More examples of PNS homogeneous polynomials can be found in [7, 26].

According to Hilbert [10, 26], two cases with low values of m and n, in which there are PNS homogeneous polynomials, are the case that $m = 6$ and $n = 3$, and the case that $m = n = 4$. In [16] and [5], sixth order three dimensional Hankel tensors and fourth order four dimensional Hankel tensors were studied respectively. No PNS Hankel tensors were found there. However, even for such low values of m and n, a strict proof to show that PNS Hankel tensors do not exist seems still very difficult. In these two cases, the Hankel tensors have thirteen independent entries of their generating vectors. Even assuming that their generating vectors are symmetric [5], there are still seven independent entries. The situation is still complicated enough to make a strict proof [5].

Are there other special classes of even order symmetric structured tensors, there are no easily checkable conditions for their positive semi-definiteness, but it is possible that they are PNS-free? A good candidate for such a problem is the class of even order strongly symmetric circulant tensors.

Strongly symmetric tensors were introduced in [25]. An mth order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called a strongly symmetric tensor if

$$a_{i_1 \cdots i_m} \equiv a_{j_1 \cdots j_m}$$

as long as $\{i_1, \cdots, i_m\} = \{j_1, \cdots, j_m\}$. Note that a symmetric matrix is a strongly symmetric tensor. Hence, strongly symmetric tensors are also extensions of symmetric matrices. Some good properties of symmetric matrices may be inherited by strongly symmetric tensors, not symmetric tensors.

Circulant tensors have applications in stochastic process and spectral hypergraph theory [6]. An mth order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called a circulant tensor if

$$a_{i_1 \cdots i_m} \equiv a_{j_1 \cdots j_m}$$

as long as $j_l = i_l + 1, mod(n)$ for $l = 1, \cdots, m$.

Hence, in this paper, we consider even order three dimensional strongly symmetric circulant tensors.

We find that a general three dimensional strongly symmetric circulant tensor has only three independent entries: the diagonal entry d, the off-diagonal entry u, which has two different indices, and the off-diagonal entry c, which has three different indices. Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an mth order three dimensional strongly symmetric circulant tensor. Here m can be even or odd. We denote

$$a_S \equiv a_{i_1 \cdots i_m}$$

if $S = \{i_1, \cdots, i_m\}$. Since $\mathcal{A}$ is strongly symmetric, it has at most seven independent entries for $S = \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$ and $\{1,2,3\}$. Since $\mathcal{A}$ is also circulant, we have

$$a_{\{1\}} = a_{\{2\}} = a_{\{3\}}, \text{ and } a_{\{1,2\}} = a_{\{2,3\}} = a_{\{1,3\}}.$$

Let $d = a_{\{1\}} = a_{\{2\}} = a_{\{3\}}$, $u = a_{\{1,2\}} = a_{\{2,3\}} = a_{\{1,3\}}$ and $c = a_{\{1,2,3\}}$. Then we see that d is the diagonal entry of $\mathcal{A}$: $d = a_{1 \dots 1} = a_{2 \dots 2} = a_{3 \dots 3}$, and an mth order three dimensional strongly symmetric circulant tensor has only three independent entries d, u and c. Thus, we may denote a general three dimensional strongly symmetric circulant tensor $\mathcal{A} = \mathcal{A}(m, d, u, c)$, where m is its order. When the context is clear, we only use $\mathcal{A}$ to denote it.

In our discussion, we need the concept of H-eigenvalues of symmetric tensors, which was introduced in [22] and is closely related to positive semi-definiteness of even order symmetric

tensors. In the next section, we introduce H-eigenvalues and discuss their relations with positive semi-definiteness of even order symmetric tensors.

Now, let $m = 2k$ be even. In Section 3, we show that there are two one-variable functions $M_c(u)$ and $N_c(u)$, such that $M_c(u) \geq N_c(u) \geq 0$, $\mathcal{A}$ is SOS if and only if $d \geq M_c(u)$, and $\mathcal{A}$ is PSD if and only if $d \geq N_c(u)$. If $M_c(u) = N_c(u)$, then three dimensional strongly symmetric PNS circulant tensors do not exist for such u and c. We show that if $u, c \leq 0$ or $u = c > 0$, then $M_c(u) = N_c(u)$. Explicit formulas for $M_c(u) = N_c(u)$ are given there in these cases. Thus, it is PNS-free for such u and c.

Note that $\mathcal{A}$ is PSD or SOS if and only if $\alpha\mathcal{A}$ is PSD or SOS, respectively. Thus, we only need to consider three cases that $c = 0$, $c = 1$ and $c = -1$.

In Section 4, we discuss the case that $c = 0$. In this case, for $u > 0$, we have $M_0(u) = uM_0(1)$ and $N_0(u) = uN_0(1)$. We show that $-N_0(1)$ is the smallest H-eigenvalue of $\mathcal{A}(m, 0, 1, 0)$. Numerical tests show that $M_0(1) = N_0(1)$ for $m = 6, 8, 10, 12$ and 14.

In Section 5, we study the case that $c = -1$. We show that there is a $u_0 > 0$ such that if $u \leq u_0$, $N_{-1}(u)$ is linear and the explicit formula of $N_{-1}(u)$ can be given, and if $u > u_0$, $N_{-1}(u)$ is the smallest H-eigenvalue of a tensor with u as a parameter. Numerical tests show that for $u > 0$, we still have $M_{-1}(u) = N_{-1}(u)$ for $m = 6, 8, 10$ and 12.

In Section 6, we study the case that $c = 1$. We show that there is a $v_0 < 0$ such that if $u \leq v_0$, $N_1(u)$ is linear and the explicit formula of $N_1(u)$ can be given, and if $u > v_0$, $N_1(u)$ is the smallest H-eigenvalue of a tensor with u as a parameter. Numerical tests show that for $u \neq 1$, we still have $M_1(u) = N_1(u)$ for $m = 6, 8, 10$ and 12.

Some final remarks are made in Section 7.

2 H-eigenvalues

H-eigenvalues of symmetric tensors were introduced in [22]. They are closely related to positive semi-definiteness of even order symmetric tensors. Let $\mathcal{T} = (t_{i_1 \cdots i_m})$ be an mth order n dimensional real symmetric tensor and $\mathbf{x} \in \Re^n$. Then $\mathcal{T}\mathbf{x}^{m-1}$ is a vector in $\Re^n$, with its ith component defined as

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i = \sum_{i_2, \cdots, i_m = 1}^{n} t_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

If there is $\mathbf{x} \in \Re^n, \mathbf{x} \neq \mathbf{0}$ and $\lambda \in \Re$ such that for $i = 1, \cdots, n$,

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i = \lambda x_i^{m-1},$$

then λ is called an H-eigenvalue of $\mathcal{T}$ and $\mathbf{x}$ is called its associated H-eigenvector. When m is even, H-eigenvalues always exist. $\mathcal{T}$ is PSD if and only if its smallest H-eigenvalue

is nonnegative [22]. From now on, we denote the smallest H-eigenvalue of $\mathcal{A}(m, d, u, c)$ as $\lambda_{\min}(m, d, u, c)$.

3 Functions $M_c(u)$ and $N_c(u)$

In this section and the next three sections, we assume that $n = 3$ and $m = 2k$ is even. Let $\mathcal{A}$ be an mth order three dimensional strongly symmetric circulant tensor. Then we may write $f_c(\mathbf{x}) \equiv f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ as

$$\begin{aligned} f_c(\mathbf{x}) &= d(x_1^m + x_2^m + x_3^m) + u \sum_{p=1}^{m-1} \binom{m}{p} (x_1^{m-p} x_2^p + x_1^{m-p} x_3^p + x_2^{m-p} x_3^p) \\ &\quad + c \sum_{p=1}^{m-2} \sum_{q=1}^{m-p-1} \binom{m}{p} \binom{m-p}{q} x_1^{m-p-q} x_2^p x_3^q. \end{aligned} \tag{1}$$

We now establish two functions $M_c(u)$ and $N_c(u)$, in the following theorem. Recall that for an mth order n dimensional tensor $\mathcal{A} = a_{i_1 \cdots 1_m}$, its ith off-diagonal entry absolute value sum is defined as

$$r_i = \sum_{i_2, \cdots, i_m = 1}^{n} |a_{i i_2 \cdots i_m}| - |a_{i i \cdots i}|.$$

If $a_{i \cdots i} \geq r_i$ for $i = 1, \cdots, n$, then $\mathcal{A}$ is called diagonally dominated, and all of its H-eigenvalues [22] are nonnegative. If furthermore $\mathcal{A}$ is even order and symmetric, then $\mathcal{A}$ is PSD [22] and SOS [3].

Theorem 1 *Let $\mathcal{A}$ be an mth order three dimensional strongly symmetric circulant tensor. Then, there are two convex functions $M_c(u) \geq N_c(u) \geq 0$ such that $\mathcal{A}$ is SOS if and only if $d \geq M_c(u)$, and $\mathcal{A}$ is PSD if and only if $d \geq N_c(u)$. If $M_c(u) = N_c(u)$, then mth order three dimensional PNS strongly symmetric circulant tensors do not exist for such u and c. Furthermore, we have*

$$M_c(u) \leq |u|(2^m - 2) + |c|(3^{m-1} - 2^m + 1). \tag{2}$$

Proof Since $\mathcal{A}$ is a circulant tensor, then it has the same off-diagonal entry absolute value sum for different rows, i.e., $r_1 = r_2 = r_3$. By (1), this row sum is equal to the right hand side of (2). Thus, if d is greater than or equal to this value, $\mathcal{A}$ is diagonally dominated and thus PSD and SOS. This shows the existence of $N_c(u)$, $M_c(u)$ and (2). As the set of PSD tensors and the set of SOS tensors are convex [17], $M_c(u)$ and $N_c(u)$ are convex. Since a necessary condition for an even order circulant tensor to be PSD is that its diagonal entry to be nonnegative [6], we have $N_c(u) \geq 0$ for all u and c.

By definition, we have $N_c(u) \le M_c(u)$. Clearly, if $M_c(u) = N_c(u)$, then mth order three dimensional PNS strongly symmetric circulant tensors do not exist for such u and c. The theorem is proved. $\square$

As discussed in the introduction, for the PNS-free problem, we only need to consider three values of c: $c = 0, 1$ and -1.

If $M_c(u) = N_c(u)$, u is called a **PNS-free point** for c.

For the convenience, we present formally three ingredients used in theoretical proofs of PNS-free points. If a point u enjoys these ingredients, it is PNS-free.

Definition 1 *Suppose that $n = 3$ and $m = 2k$ is even. Suppose that there is a number M such that $\mathcal{A}$ is SOS if $d = M$, and a nonzero vector $\bar{\mathbf{x}} \in \Re^3$ such that $f_c^*(\bar{\mathbf{x}}) = 0$, where $f_c^*(\mathbf{x}) \equiv f_c(\mathbf{x})$ with $d = M$. Then we call M the* **critical value** *of $\mathcal{A}$ at u, the SOS decomposition $f_c^*(\mathbf{x})$ the* **critical SOS decomposition** *of $\mathcal{A}$ at P, and $\bar{\mathbf{x}}$ the* **critical minimizer** *of $\mathcal{A}$ at u.*

Theorem 2 *Let $u \in \Re$. Then u is PNS-free for c if $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f_c^*(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at u.*

Proof Suppose that $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f_c^*(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at u. Then we have $M \ge M_c(u)$ by the definition of $M_c(u)$. If $d < M$, then

$$f_c(\bar{\mathbf{x}}) = (d - M)(\bar{x}_1^m + \bar{x}_2^m + \bar{x}_3^m) + f_c^*(\bar{\mathbf{x}}) < 0.$$

This implies that $N_c(u) \ge M$ by the definition of $N_c(u)$. But $N_c(u) \le M_c(u)$. Thus, $M_c(u) = N_c(u) = M$, i.e., u is PNS-free for c. $\square$

Corollary 1 *If $u, c \le 0$, then*

$$M_c(u) = N_c(u) = -u(2^m - 2) - c(3^{m-1} - 2^m + 1). \tag{3}$$

Thus, it is PNS-free for such u and c.

Proof Suppose that $u, c \le 0$. Let M be the value of the right hand side of (2), and $\bar{\mathbf{x}} = (1, 1, 1)^\top$. If $d = M$, then $f_c(\mathbf{x}) = f_c^*(\mathbf{x})$ has an SOS decomposition as $\mathcal{A}$ is an even order diagonally dominated symmetric tensor [3]. We also see that $f_c^*(\bar{\mathbf{x}}) = 0$. The result follows. $\square$

Corollary 2 *If $u = c > 0$, then*

$$M_c(u) = N_c(u) = u = c.$$

Thus, it is PNS-free for such u and c.

Proof Suppose that $u = c > 0$. Let $M = u = c$, and $\bar{\mathbf{x}} = (2, -1, -1)^\top$. If $d = M$, then $f_c(\mathbf{x}) = f_c^*(\mathbf{x}) = (x_1 + x_2 + x_2)^m$ has an SOS decomposition. We also see that $f_c^*(\bar{\mathbf{x}}) = 0$. The result follows. $\square$

Corollary 3 *If $u > 0$, then*

$$M_0(u) = uM_0(1)$$

and

$$N_0(u) = uN_0(1).$$

Hence, for $c = 0$, it is PNS-free if and only if $M_0(1) = N_0(1)$.

Proof Suppose that $u > 0$ and $d \geq uM_0(1)$. By (1), we have

$$\begin{aligned} f_0(\mathbf{x}) &= (d - uM_0(1))(x_1^m + x_2^m + x_3^m) \\ &\quad + u\left(M_0(1)(x_1^m + x_2^m + x_3^m) + \sum_{p=1}^{m-1} \binom{m}{p} (x_1^{m-p}x_2^p + x_1^{m-p}x_3^p + x_2^{m-p}x_3^p)\right). \end{aligned}$$

We see that $f_0(\mathbf{x})$ is SOS. Hence, $M_0(u) = uM_0(1)$. Similarly, we may prove that $N_0(u) = uN_0(1)$. By these and Corollary 1, we have the last conclusion. $\square$

4 $c = 0$

If $u \leq 0$, by Corollary 1, we have $M_0(u) = N_0(u) = -u(2^m - 2)$. If $u > 0$, by Corollary 3, we have $M_0(u) = uM_0(1)$ and $N_0(u) = uN_0(1)$. We only need to consider the case that $u = 1$.

Proposition 1 *We have that $N_0(1) = -\lambda_{\min}(m, 0, 1, 0)$.*

Proof By [22], $\mathcal{A}(m, d, 1, 0)$ is PSD if and only if $\lambda_{\min}(m, d, 1, 0) \geq 0$. By the structure of circulant tensors, $\lambda_{\min}(m, d, 1, 0) = d + \lambda_{\min}(m, 0, 1, 0)$. Thus, $\mathcal{A}(m, d, 1, 0)$ is PSD if and only if $d \geq -\lambda_{\min}(m, 0, 1, 0)$. By the definition of $N_c(u)$, we have $N_0(1) = -\lambda_{\min}(m, 0, 1, 0)$. $\square$

For $m = 6, 8, 10, 12$ and 14, we compute $M_0(1)$ and $N_0(1)$ by using Matlab (YALMIP, GloptiPloy and SeDuMi) software and Maple [9, 18, 19, 28], respectively. We find for such m, $M_0(1) = N_0(1)$. The results are displayed in Table 1.

m	$M_0(1)$	$N_0(1)$
6	1.737348471173345	1.737348471777547
8	1.882980354978972	1.882980356780414
10	1.947977161918168	1.947977172341075
12	1.976878006619490	1.976878047128592
14	1.989722829997529	1.989723542124766

Table 1: The values of $M_0(1)$ and $N_0(1)$.

5 $c = -1$

If $u \leq 0$, then Corollary 1 indicates that $M_{-1}(u) = N_{-1}(u) = -u(2^m-2)+(3^{m-1}-2^m+1)$. We now discuss the case that $u > 0$.

In this section and the next section, we denote that $\mathcal{B} = \mathcal{A}(m, 3^{m-1} - 2^m + 1, 0, -1)$ and $\mathcal{T} = \mathcal{A}(m, 2^m - 2, -1, 0)$. Then, $\mathcal{B}$ and $\mathcal{T}$ are obviously diagonally dominated. Hence, they are PSD and SOS [3]. And all of their H-eigenvalues are nonnegative.

Theorem 3 *Let*

$$\varphi(u) \equiv \lambda_{\min}(\mathcal{B} - u\mathcal{T}),$$

where $\lambda_{\min}(\cdot)$ denotes the smallest H-eigenvalue. Then, $\varphi(u) \leq 0$. If $\varphi(u) = 0$, then we have

$$N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2). \tag{4}$$

If $\varphi(u) < 0$, then we have

$$N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2) - \lambda_{\min}(m, 3^{m-1} - 2^m + 1 - u(2^m - 2), u, -1). \tag{5}$$

Furthermore, the set $C = \{u : \varphi(u) = 0\}$ is a nonempty closed convex ray $(-\infty, u_0]$ for some $u_0 \geq 0$.

Proof Let $\bar{x} = (1, 1, 1)^\top$. Then $\mathcal{B}\bar{\mathbf{x}}^m = 0$ and $\mathcal{T}\bar{\mathbf{x}}^m = 0$. Thus, $(\mathcal{B} - u\mathcal{T})\bar{\mathbf{x}}^m = 0$ for any u. By [22], we see that

$$\varphi(u) \equiv \lambda_{\min}(\mathcal{B} - u\mathcal{T}) \leq 0.$$

If $\varphi(u) = 0$, let $d = 3^{m-1} - 2^m + 1 - u(2^m - 2)$. Then $\mathcal{A}(m, d, u, -1) = \mathcal{B} - u\mathcal{T}$. We have $\lambda_{\min}(m, d, u, -1) = 0$. This implies (4).

If $\varphi(u) < 0$, then, because

$$f_{-1}(\mathbf{x}) = (d - (3^{m-1} - 2^m + 1) + u(2^m - 2))\mathcal{I}\mathbf{x}^m + \mathcal{B}\mathbf{x}^m - u\mathcal{T}\mathbf{x}^m \geq 0,$$

we have

$$
\begin{aligned}
N_{-1}(u) &= \inf\{d \ : \ \lambda_{\min}((d-(3^{m-1}-2^m+1)+u(2^m-2))\mathcal{I}+\mathcal{B}-u\mathcal{T}) \geq 0\} \\
&= (3^{m-1}-2^m+1)-u(2^m-2)-\lambda_{\min}(\mathcal{B}-u\mathcal{T}) \\
&= 3^{m-1}-2^m+1-u(2^m-2)-\lambda_{\min}(m, 3^{m-1}-2^m+1-u(2^m-2), u, -1).
\end{aligned}
$$

We have (5).

By Corollary 1, C is nonempty and $u \in C$ as long as $u \leq 0$. By Theorem 1, $N_{-1}(u)$ is a convex function. By this, (4) and (5), C is convex. Since $\lambda_{\min}$ is a continuous function [22], C is closed. Since $u \in C$ as long as $u \leq 0$, C is a ray, with the form $(-\infty, u_0]$ for some $u_0 \geq 0$. $\square$

Corollary 4 *Let $u_0 \equiv \max\{\hat{u} \ : \ \varphi(\hat{u}) = 0\}$. Then u_0 is well-defined and $u_0 \geq 0$. Furthermore, for $u \leq u_0$, we have (4), and for $u > u_0$, we have (5).*

Proposition 2 *If $M_{-1}(u_0) = N_{-1}(u_0) = 3^{m-1}-2^m+1-u_0(2^m-2)$, then for $u \leq u_0$, we have $M_{-1}(u) = N_{-1}(u) = 3^{m-1}-2^m+1-u(2^m-2)$.*

Proof By Theorem 1, $M_{-1}(u)$ is convex. By Corollary 1, $M_{-1}(u) = 3^{m-1}-2^m+1-u(2^m-2)$ for $u \leq 0$. Since $u_0 \geq 0$, the conclusion follows. $\square$

Proposition 3 *Suppose $u_0 = \max\{\hat{u} \ : \ \varphi(\hat{u}) = 0\}$. Then, we have*

$$
0 \leq u_0 \leq \bar{u}_0(m) \equiv \frac{3^{m-1}+1}{2^m} - 1. \tag{6}
$$

Proof Since $\mathcal{B}$ is PSD and has a H-eigenvalue 0, we have $\varphi(0) = 0$ and $u_0 \geq 0$.

On the other hand, we consider the case $u > \bar{u}_0$. Let $\mathbf{x}_0 = (1, 1, -3)^\top$. We have

$$
(\mathcal{B}-\bar{u}_0\mathcal{T})\mathbf{x}_0^m = 0 \qquad \text{and} \qquad \mathcal{T}\mathbf{x}_0^m = 2^m(3^m-1).
$$

Then,

$$
(\mathcal{B}-u\mathcal{T})\mathbf{x}_0^m = (\mathcal{B}-\bar{u}_0\mathcal{T})\mathbf{x}_0^m - (u-\bar{u}_0)\mathcal{T}\mathbf{x}_0^m = -(u-\bar{u}_0)2^m(3^m-1) < 0.
$$

Hence, we have $\varphi(u) = \lambda_{\min}(\mathcal{B}-u\mathcal{T}) < 0$ when $u > \bar{u}_0$. Therefore, $u_0 \leq \bar{u}_0$. $\square$

For $m = 6, 8, 10, 12$ and 14, we find that $\mathcal{B}-\bar{u}_0\mathcal{T}$ is PSD. This shows that for such m, $\varphi(\bar{u}_0) = 0$, i.e.,

$$
u_0 = \bar{u}_0(m) \equiv \frac{3^{m-1}+1}{2^m} - 1. \tag{7}
$$

It remains a further research topic to show that $\mathcal{B}-\bar{u}_0\mathcal{T}$ is PSD for all even m with $m \geq 16$. If this is true, then (7) is true for all even m with $m \geq 6$.

In Tables 2-5, the values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 6, 8, 10, 12$ and $u = 0.1, 2, \frac{45}{16}, 5, 10, 40, 300$, $u = 1, 3, \frac{483}{64}, 10, 40, 140, 300$, $u = 1, 10, \frac{4665}{256}, 20, 40, 140, 300$ and $u = 1, 20, \frac{43263}{1024}, 60, 100, 140, 300$ are reported, respectively. We find for such m and u, $M_{-1}(u) = N_{-1}(u)$.

u	$M_{-1}(u)$	$N_{-1}(u)$
0.1	173.799999999899	173.8
2	55.9999999995172	56
$\frac{45}{16}$	5.62499991033116	5.625
5	9.42544641511067	9.4254465011842588
10	18.1121860822789	18.112186280892696
40	70.2326321651344	70.232638183914150
300	521.943237017699	521.94324013633004

Table 2: The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 6$.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	1677.99999992219	1678
3	1169.99999999356	1170
$\frac{483}{64}$	15.0937478786308	15.09375
10	19.7129359300341	19.7129361640501
40	76.2023466001335	76.2023468071730
140	264.500365037152	264.500382469583
300	565.777184078832	565.777239551091

Table 3: The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 8$.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	17637.9999999549	17638
10	8439.99999783081	8440
$\frac{4665}{256}$	36.4452603485520	36.4453125
20	39.9075358817909	39.9075375522954
40	78.8670625326286	78.8670809985488
140	273.664775923815	273.664798232238
300	585.341085323688	585.341145806726

Table 4: The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 10$.

6 $c = 1$

Corollary 2 indicates that $M_1(1) = N_1(1) = 1$. Hence, we only need to consider the case that $u \neq 1$. Let $\mathcal{B}$ and $\mathcal{T}$ be the same as in the last section. We have the following theorem.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	168957.999979042	168958
20	91171.9999996683	91172
$\frac{43263}{1024}$	84.4971787852022	84.498046875
60	119.589505579120	119.589562756497
100	198.664532858285	198.664684641639
140	277.739708996851	277.739806526784
300	594.040191670531	594.040294067366

Table 5: The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 12$.

Theorem 4 *Let*

$$\psi(u) \equiv \lambda_{\min}(-u\mathcal{T} - \mathcal{B}).$$

Then, $\psi(u) \leq 0$. If $\psi(u) = 0$, then we have

$$N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2). \tag{8}$$

If $\psi(u) < 0$, then we have

$$N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2) - \lambda_{\min}(m, -(3^{m-1} - 2^m + 1) - u(2^m - 2), u, 1). \tag{9}$$

Furthermore, if the set $C = \{u : \psi(u) = 0\}$ is nonempty, then it is a closed convex ray $(-\infty, v_0]$ for some $v_0 < 0$.

Proof The proof of this theorem is similar to the proof of Theorem 3. However, we cannot apply Corollary 1 here. If C is non empty, we may show that C is closed and convex as in the proof of Theorem 3.

If there is a $\hat{u} \leq 0$ such that $\lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B}) = 0$, for $u \leq \hat{u} \leq 0$, we have

$$\begin{aligned}
\psi(u) &= \lambda_{\min}(-u\mathcal{T} - \mathcal{B}) \\
&= \lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B} + (-u + \hat{u})\mathcal{T}) \\
&\geq \lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B}) \\
&= 0.
\end{aligned}$$

Hence, $\psi(u) = 0$ for all $u \leq \hat{u} \leq 0$. So if C is not empty, it is a ray with the form $(-\infty, v_0]$ for some v_0. Clearly, $\psi(0) \equiv \lambda_{\min}(-\mathcal{B}) < 0$ as $\mathcal{B}$ is PSD and not a zero tensor. Hence, $v_0 < 0$.

The other parts of the proof are similar to the proof of Theorem 3. □

Corollary 5 *If there is one point $\hat{u}$ such that $\psi(\hat{u}) = 0$, let $v_0 \equiv \max\{\hat{u} : \psi(\hat{u}) = 0\}$. Then for $u \leq v_0$, we have (8), and for $u \geq v_0$, we have (9).*

We also have the following proposition.

Proposition 4 *If $M_1(v_0) = N_1(v_0) = -(3^{m-1} - 2^m + 1) - v_0(2^m - 2)$, then for $u \le v_0$, we have $M_1(u) = N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$.*

Proof Suppose that $M_1(v_0) = N_1(v_0) = -(3^{m-1} - 2^m + 1) - v_0(2^m - 2)$. By (3), if $u \le v_0$ and $d = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$, we have

$$f_1^*(\mathbf{x}) = -\bar{u} g_1(\mathbf{x}) + g_2(\mathbf{x}),$$

where

$$g_1(\mathbf{x}) = \mathcal{A}(m, 2^m - 2, -1, 0),$$
$$g_2(\mathbf{x}) = \mathcal{A}(m, -(3^{m-1} - 2^m + 1) - v_0(2^m - 2), v_0, 1)$$

and

$$\bar{u} \equiv u - v_0 \le 0.$$

We see that $g_2(\mathbf{x})$ is equal to the critical SOS decomposition of $\mathcal{A}$ at $c = 1$ and $u = v_0$, and $g_1(\mathbf{x})$ is equal to the critical SOS decomposition of $\mathcal{A}$ at $c = 0$ and $u = -1$. Hence both $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ are SOS polynomials. This implies that $f_1^*(\mathbf{x})$ is an SOS polynomial. Let $\bar{\mathbf{x}} = (1, 1, 1)^\top$, we see that $f_1^*(\bar{\mathbf{x}}) = 0$. Then the conclusion follows from Theorem 2. $\square$

Proposition 5 *Suppose that C is not empty. Let $v_0 = \max\{\hat{u} \ : \ \psi(\hat{u}) = 0\}$. Then, we have*

$$v_0 \le \bar{v}_0(m) \equiv 1 - \frac{3^{m-1}}{2^{m-1} + 1}. \tag{10}$$

Proof Let $\mathbf{x}_0 = (1, 1, -\frac{1}{2})^\top$. We have

$$(-\mathcal{B} - \bar{v}_0 \mathcal{T})\mathbf{x}_0^m = 0 \qquad \text{and} \qquad \mathcal{T}\mathbf{x}_0^m = 2^m + 1 - 2^{1-m}.$$

Then,

$$(-\mathcal{B} - u\mathcal{T})\mathbf{x}_0^m = (-\mathcal{B} - \bar{v}_0\mathcal{T})\mathbf{x}_0^m - (u - \bar{v}_0)\mathcal{T}\mathbf{x}_0^m = -(u - \bar{v}_0)(2^m + 1 - 2^{1-m}) < 0.$$

Hence, we have $\psi(u) = \lambda_{\min}(-\mathcal{B} - u\mathcal{T}) < 0$ when $u > \bar{v}_0$. Therefore, $v_0 \le \bar{v}_0$. $\square$

Similar to the discussion of u_0, for $m = 6, 8, 10, 12$ and 14, we find that $-\bar{v}_0\mathcal{T} - \mathcal{B}$ is PSD. This shows that for such m, $\psi(\bar{v}_0) = 0$, i.e.,

$$v_0 = \bar{v}_0(m) \equiv 1 - \frac{3^{m-1}}{2^{m-1} + 1}. \tag{11}$$

This also shows that C is not empty for such m. It remains a further research topic to show that $-\bar{v}_0\mathcal{T} - \mathcal{B}$ is PSD for all even m with $m \ge 16$. If this is true, then (11) is true for all even m with $m \ge 6$.

In Tables 6-9, the values of $M_1(u)$ and $N_1(u)$ for $m = 6, 8, 10, 12$ and $u = -40, -10, -\frac{70}{11}, -5, -1, 10, 40$, $u = -40, -20, -\frac{686}{43}, -10, -1, 10, 40$, $u = -100, -40, -\frac{710}{19}, -20, -1, 10, 40$ and $u = -140, -100, -\frac{58366}{683}, -60, -1, 10, 40$ are reported, respectively. We find for such m and u, $M_1(u) = N_1(u)$.

u	$M_1(u)$	$N_1(u)$
-40	2299.99999993444	2300
-10	439.999999987168	440
$-\frac{70}{11}$	214.545454213196	214.54545454545454
-5	173.991050854352	173.99105151869704
-1	55.8846973214056	55.884697712412670
10	16.6347897201042	16.634789948247836
40	68.7552353830704	68.755241242186800

Table 6: The values of $M_1(u)$ and $N_1(u)$ for $m = 6$.

u	$M_1(u)$	$N_1(u)$
-40	8228.00000000754	8228
-20	3147.99999997053	3148
$-\frac{686}{43}$	2120.18604151092	2120.18604651163
-10	1371.80748977461	1371.80749709544
-1	243.740078469126	243.740080311110
10	17.9466697015668	17.9466711544215
40	74.4360734714431	74.4360817805826

Table 7: The values of $M_1(u)$ and $N_1(u)$ for $m = 8$.

u	$M_1(u)$	$N_1(u)$
-100	83539.9999994888	83540
-40	22219.9999995661	22220
$-\frac{710}{19}$	19530.5255392283	19530.5263157895
-20	10678.1156381743	10678.1156702343
-1	1004.40451207948	1004.40454284172
10	18.5317674915799	18.5317776218259
40	76.9710868541002	76.9710927899669

Table 8: The values of $M_1(u)$ and $N_1(u)$ for $m = 10$.

7 Final Remarks

In Proposition 1, Theorems 3 and 4, we give a necessary and sufficient condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite. For $u, c \leq 0$ and $u = c > 0$, we show that this condition is also sufficient for this tensor to be sum-of-squares. Numerical tests indicate that this is also true in the other

u	$M_1(u)$	$N_1(u)$
-140	400107.999992756	400108
-100	236347.999998551	236348
$-\frac{58366}{683}$	176802.173593347	176802.170881802
-60	124727.840916646	124727.840144917
-1	4063.38103939314	4063.38106552746
10	18.7918976770375	18.7919005425937
40	78.0982265029468	78.0982419563963

Table 9: The values of $M_1(u)$ and $N_1(u)$ for $m = 12$.

cases.

How can $\mathcal{B} - \bar{u}_0\mathcal{T}$ and $-\bar{v}_0\mathcal{T} - \mathcal{B}$ be shown to be PSD for all even $m \geq 6$? If these are true, then (7) and (11) are true for all even $m \geq 6$.

Finally, more efforts are needed to prove that this problem is PNS-free eventually.

References

[1] A.A. Ahmadi and P.A. Parrilo, "A convex polynomial that is not sos-convex", *Mathematical Programming* **135** (2012) 275-292.

[2] R. Badeau and R. Boyer, "Fast multilinear singular value decomposition for structured tensors", *SIAM Journal on Matrix Analysis and Applications* **30** (2008) 1008-1021.

[3] H. Chen, G. Li and L. Qi, "Sum-of-Squares tensors and their sum-of-squares ranks", April 2015. arXiv:1504.03414.

[4] H. Chen and L. Qi, "Positive definiteness and semi-definiteness of even order symmetric Cauchy tensors", *Journal of Industrial Management and Optimization* **11** (2015) 1263-1274.

[5] Y. Chen, L. Qi and Q. Wang, "Positive semi-definiteness and sum-of-squares property of fourth order four dimensional Hankel tensors", March 2015. arXiv:1502.04566.

[6] Z. Chen and L. Qi, "Circulant tensors with applications to spectral hypergraph theory and stochastic process", November 2014. arXiv:1312.2752

[7] M.D. Choi and T.Y. Lam, "Extremal positive semidefinite forms", *Mathematische Annalen* **231** (1977) 1-18.

[8] W. Ding, L. Qi and Y. Wei, "Fast Hankel tensor-vector products and application to exponential data fitting", *Numerical Linear Algebra with Applications*, (2015), DOI: 10.1002/nla.1970.

[9] D. Henrion, J.B. Lasserre and J. Löfberg, "GloptiPoly 3: moments, optimization and semidefinite programming", *Optimization Methods and Software* **24** (2009) 761-779.

[10] D. Hilbert, "Über die Darstellung definiter Formen als Summe von Formenquadraten", *Mathematical Annals,* **32** (1888) 342-350.

[11] M.R. Kannan, N. Shaked-Monderer and A. Berman, "Some properties of strong H-tensors and general H-tensors", *Linear Algebra and its Applications* **476** (2015) 42-55.

[12] J.B. Lasserre, "Global optimization with polynomials and the problem of moments", *SIAM Journal on Optimization* **11** (2001) 796-817.

[13] M. Laurent, "Sums of squares, moment matrices and optimization over polynomials", *Emerging Applications of Algebraic Geometry*, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant eds., Springer, (2009) pp. 157-270.

[14] C. Li and Y. Li, "Double B-tensors and quasi-double B-tensors", *Linear Algebra and Its Applications* **21**, (2014) 39-50.

[15] C. Li, F. Wang, J. Zhao, Y. Zhu and Y. Li, "Criterions for the positive definiteness of real supersymmetric tensors", *Journal of Computational and Applied Mathematics* **255** (2014) 1-14.

[16] G. Li, L. Qi and Q. Wang, "Are there sixth order three dimensional Hankel tensors?", arXiv:1411.2368, November 2014.

[17] G. Li, L. Qi and Y. Xu, "SOS-Hankel Tensors: Theory and Application", arXiv:1410.6989, October 2014.

[18] J. Löfberg, "YALMIP: A Toolbox for Modeling and Optimization in MATLAB", In: *Proceedings of the CACSD Conference*, Taipei, Taiwan, (2004).

[19] J. Löfberg, "Pre- and post-processing sum-of-squares programs in practice", *IEEE Transactions on Automatic Control* **54** (2004) 1007-1011.

[20] T.S. Motzkin, “The arithmetic-geometric inequality”, In: *Inequalities*, O. Shisha ed., Academic Press, New York, (1967) pp. 205-224.

[21] J.M. Papy, L. De Lauauwer and S. Van Huffel, “Exponential data fitting using multilinear algebra: The single-channel and multi-channel case”, *Numerical Linear Algebra with Applications* **12** (2005) 809-826.

[22] L. Qi, “Eigenvalues of a real supersymmetric tensor”, *Journal of Symbolic Computation* **40** (2005) 1302-1324.

[23] L. Qi, “Hankel tensors: Associated Hankel matrices and Vandermonde decomposition”, *Communications in Mathematical Sciences* **13** (2015) 113-125.

[24] L. Qi and Y. Song, “An even order symmetric B tensor is positive definite”, *Linear Algebra and Its Applications* **457** (2014) 303-312.

[25] L. Qi, C. Xu and Y. Xu, “Nonnegative tensor factorization, completely positive tensors and a hierarchically elimination algorithm”, *SIAM Journal on Matrix Analysis and Applications* **35** (2014) 1227-1241.

[26] B. Reznick, “Some concrete aspects of Hilbert’s 17th problem”, *Contemporary Mathematics* **253** (2000) 251-272.

[27] Y. Song and L. Qi, “Infinite and finite dimensional Hilbert tensors”, *Linear Algebra and Its Applications* **451** (2014) 1-14.

[28] J.F. Sturm, “SeDuMi version 1.1R3”, (2006). Available at http://sedumi.ie.lehigh.edu.

[29] C. Xu, “Hankel tensors, Vandermonde tensors and their positivities”, *Linear Algebra and Its Applications*, (2015), doi:10.1016/j.laa.2015.02.012.

[30] L. Zhang, L. Qi and G. Zhou, “M-tensors and some applications”, *SIAM Journal on Matrix Analysis and Applications* **35** (2014) 437-452.